On Non-Linear Magnetohydrodynamic Flow due to Peristaltic Transport of an Oldroyd 3-Constant Fluid

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In this work a theoretical analysis is presented for the problem of peristaltic transport of an incompressible Oldroyd 3-constant fluid in an infinite channel with flexible walls. The flow is induced by an infinite sinusoidal wave train moving along the walls of the channel. The fluid is electrically conducting and a magnetic field has been applied transversely to the flow. This problem has numerous applications in various branches of science. A perturbation solution of the stream function for zeroth-, first- and second-order in a small amplitude ratio is obtained. The obtained results are illustrated graphically to show salient features of the solutions. The effect of the magnetic parameter, the relaxation time and the retardation time on the mean axial velocity and the reversal flow is investigated. It is found that the possibility of flow reversal increases by increasing the magnetic parameter and viscoelastic parameters. The results show that the values of the mean axial velocity of an Oldroyd 3-constant fluid are less than these for a Newtonian fluid. Numerical results are reported for various values of the physical parameters of interest. – Mathematics Subject Classification: 76Z05.

Key words: Peristaltic Transport; Magnetohydrodynamic; Oldroyd 3-Constant Fluid; Non-Newtonian Fluid.

1. Introduction

The problem of the mechanism of peristaltic transport has attracted the attention of many investigators since the first investigation of Latham [1]. Peristalsis is a kind of fluid transport induced by a progressive wave of area contraction or expansion along the walls of a distensible duct containing liquid. This kind of fluid transport appears in many biological organs such as lower intestine, gastrointestinal tract, cervical canal, female fallopian tube, lymphatic vessels and small blood vessels. Also, peristaltic transport occurs in many practical applications involving biomechanical systems such as roller and finger pumps. Jaffrin and Shapiro [2] presented a review of much of the early literature. They explained the basic principles of peristaltic pumping and brought out clearly the significance of various parameters governing the flow. A summary of most of the investigations, reported up to the year 1983, has been presented by Srivastava and Srivastava [3]. The important contributions of recent years to the topic are referenced in the literature [4 – 7]. Most theoretical investigations have been carried out for Newtonian fluids, although it is known that most physiological fluids behave like non-Newtonian fluids. In this regard there is only limited information on the transport of non-Newtonian fluids. The main reason is that additional non-linear terms appear in the equations of motion rendering the problem more difficult to solve. Another reason is that a universal non-Newtonian constitutive relation that can be used for all fluids and flows is not available. The earliest ones date back to Raju and Devanathan [8, 9]. They considered the motion of an inelastic power-law fluid and of a special viscoelastic fluid of differential type of grade two through a tube with sinusoidal corrugation of small amplitude propagating in the axial direction. Bohme and Friedrich [10] have investigated the peristaltic flow of viscoelastic liquids under assumptions that the relevant Reynolds number is small enough to neglect inertia forces, and that the ratio of the wave length and the channel height is large, which implies that the pressure is constant over the cross-section. Mernone et al. [11] have considered the peristaltic flow of rheologically complex physiological fluids when modelled by a non-Newtonian Casson fluid in a two-dimensional channel. Misra and Pandey [12] have studied the peristaltic flow of blood in small vessels by developing a mathematical model in which blood has been treated as a two-layer fluid. Various authors [13 – 16] considered an Oldroyd

3-constant model in the absence of peristaltic motion. The study of hydromagnetics goes back to Faraday who predicted induced currents in the ocean due to the earth's magnetic field. The presented analysis is of interest because the theoretical study of magnetohydrodynamic channel flows has widespread applications in designing cooling systems with liquid metals, magnetohydrodynamic generators, accelerators and in the movement of conductive physiological non-Newtonian fluids, e. g., the blood and blood pump machines. The influence of a magnetic field may be utilized as a blood pump in carrying out cardiac operations for the flow of blood in arteries with arterial disease like arterial stenosis or arteriosclerosis. Another important field of application is the electromagnetic propulsion. Basically, an electromagnetic propulsion system consists of a power source, such as a nuclear reactor, a plasma, and a tube through which the plasma is accelerated by electromagnetic forces.

The aim of the present paper is to study the effect of the magnetic parameter, the relaxation time and the retardation time on peristaltic transport of an incompressible Oldroyd viscoelastic electrically conducting fluid. Such work seems to be important and useful because attention has hardly been given to the study of Oldroyd fluids. Also, some non-Newtonian models take into account normal stress differences and shear-thinning/thickening effects, but lack other features such as stress relaxation. In our analysis, we assumed that the velocity components and the pressure gradient could be expanded in a regular perturbation series of the amplitude ratio. The non-linearity of the equations of motion is taken into account. As the magnetic parameter, the relaxation time and the retardation time tend to zero, the analytical results reduce to the well-known case of a Newtonian fluid in agreement with Fung and Yih [17].

2. Basic Equations and Formulation of the Problem

We consider a two-dimensional channel of uniform width $2d$ filled with an incompressible Oldroyd viscoelastic electrically conducting fluid. We assume an infinite wave train travelling with velocity c along the walls (see Fig. 1). The continuity equation, the equation of motion and the Maxwell equations governing the flow of a magnetohydrodynamic incompressible Oldroydian fluid are

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

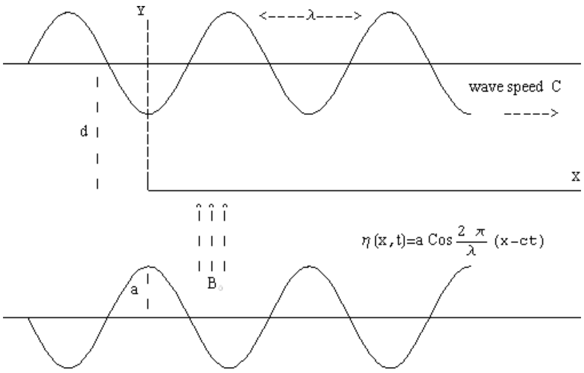

Fig. 1. Geometry of the problem.

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \operatorname{div} \boldsymbol{\Sigma} + \mathbf{J} \times \mathbf{B}, \tag{2}$$

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{B} = \mu_m \mathbf{J}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

where $\mathbf{V} = (u(x,y,t), v(x,y,t), 0)$ is the velocity field, $\boldsymbol{\Sigma}$ the Cauchy stress tensor, $\mathbf{J}$ the current density, $\mathbf{B}$ the total magnetic field, $\mathbf{E}$ the total electric field, μ_m the electric permeability and ρ the density. The generalized Ohm's law is

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{4}$$

where σ is the electrical conductivity. It is assumed, following [18, 19], that there is no applied or polarization voltage, so that $\mathbf{E} = 0$. Now we assume that a magnetic field $\mathbf{B} = (0, B_0, 0)$ with a constant magnetic flux density B_0 is applied in the y-direction. Regardless of the induced magnetic field, it follows from (4) that the magnetohydrodynamic force is

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 u \mathbf{i}. \tag{5}$$

According to Oldroyd [20], the Cauchy stress tensor $\boldsymbol{\Sigma}$ for an Oldroyd 3-constant fluid is

$$\boldsymbol{\Sigma} = -P\mathbf{I} + \mathbf{S}, \tag{6}$$

where $-P\mathbf{I}$ is the spherical part of the stress due to the constraint of incompressibility. The extra stress tensor $\mathbf{S}$ is defined by

$$\mathbf{S} + \lambda_1 \left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{LS} - \mathbf{SL}^T \right) = \mu \left(\mathbf{A}_1 + \lambda_2 \left(\frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} - \mathbf{LA}_1 - \mathbf{A}_1\mathbf{L}^T \right) \right), \tag{7}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ is the material time derivative, μ the viscosity, λ_1 and λ_2 are the material time constants referred to as relaxation and retardation time, respectively. The tensor $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{8}$$

where $\mathbf{L}$ is the spatial velocity gradient defined by $\mathbf{L} = \text{grad}\,\mathbf{V}$. It is assumed that $\lambda_1 \geq \lambda_2 \geq 0$. It should be noted that this model includes the viscous Navier-Stokes fluid as a special case for $\lambda_1 = \lambda_2 = 0$. Further, if $\lambda_2 = 0$ it reduces to a Maxwell fluid. For an unsteady two-dimensional flow we find that (1)–(8) take the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} - \sigma B_0^2\, u, \tag{10}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}, \tag{11}$$

$$S_{xx} + \lambda_1\left(\frac{\partial S_{xx}}{\partial t} + u\frac{\partial S_{xx}}{\partial x} + v\frac{\partial S_{xx}}{\partial y} - 2S_{xx}\frac{\partial u}{\partial x} - 2S_{xy}\frac{\partial u}{\partial y}\right) = 2\mu\left[\frac{\partial u}{\partial x} + \lambda_2\left(\frac{\partial^2 u}{\partial t\partial x} + u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} - 2(\frac{\partial u}{\partial x})^2 - \frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)\right], \tag{12}$$

$$S_{xy} + \lambda_1\left(\frac{\partial S_{xy}}{\partial t} + u\frac{\partial S_{xy}}{\partial x} + v\frac{\partial S_{xy}}{\partial y} - S_{xy}\frac{\partial u}{\partial x} - S_{yy}\frac{\partial u}{\partial y} - S_{xx}\frac{\partial v}{\partial x} - S_{xy}\frac{\partial v}{\partial y}\right) = \mu\Big[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \lambda_2\Big[\frac{\partial^2 u}{\partial t\partial y} + \frac{\partial^2 v}{\partial t\partial x} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\Big]\Big], \tag{13}$$

$$S_{yy} + \lambda_1\left(\frac{\partial S_{yy}}{\partial t} + u\frac{\partial S_{yy}}{\partial x} + v\frac{\partial S_{yy}}{\partial y} - 2S_{xy}\frac{\partial v}{\partial x} - 2S_{yy}\frac{\partial v}{\partial y}\right) = 2\mu\left[\frac{\partial v}{\partial y} + \lambda_2\left(\frac{\partial^2 v}{\partial t\partial y} + u\frac{\partial^2 v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} - 2\left(\frac{\partial v}{\partial y}\right)^2 - \frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)\right]. \tag{14}$$

Let the vertical displacements of the upper and lower walls be η and $-\eta$, respectively. The geometry of the wall surface is defined as

$$\eta = a\cos\frac{2\pi}{\lambda}(x - c\,t), \tag{15}$$

where a is the amplitude, λ the wave length and c the wave speed. The horizontal displacement will be assumed to be zero. Hence the boundary conditions for the fluid are

$$u = 0 \quad \text{and} \quad v = \pm\frac{\partial \eta}{\partial t} \quad \text{at} \quad y = \pm d \pm \eta. \tag{16}$$

We introduce non-dimensional variables and parameters as follows: $x^* = \frac{x}{d}, y^* = \frac{y}{d}, u^* = \frac{u}{c}, v^* = \frac{v}{c}, t^* = \frac{ct}{d}, p^* = \frac{p}{\rho c^2}, \eta^* = \frac{\eta}{d}, S^*_{xx} = \frac{d\,S_{xx}}{\mu c}, S^*_{xy} = \frac{d\,S_{xy}}{\mu c}, S^*_{yy} = \frac{d\,S_{yy}}{\mu c}$, amplitude ratio $\varepsilon = \frac{a}{d}$, wave number $\alpha = \frac{2\pi d}{\lambda}$, Reynolds number $R = \frac{cd\rho}{\mu}$, magnetic parameter $M^2 = \frac{d\sigma B_0^2}{\rho c}$ and Weissenberg numbers $w_1 = \frac{c\lambda_1}{d}$, $w_2 = \frac{c\lambda_2}{d}$.

In terms of the stream function $\psi(x,y,t)$, after eliminating P and dropping the star over the symbols, (10)–(16) become:

$$\frac{\partial}{\partial t}\nabla^2\psi + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = \frac{1}{R}\left[S_{xx,xy} + S_{xy,yy} - S_{xy,xx} - S_{yy,yx}\right] - M^2\psi_{yy}, \tag{17}$$

$$S_{xx} + w_1\left[S_{xx,t} + \psi_y S_{xx,x} - \psi_x S_{xx,y} - 2\psi_{xy}S_{xx} - 2\psi_{yy}S_{xy}\right] = 2\left[\psi_{xy} + w_2\left[\psi_{xyt} + \psi_y\psi_{yxx} - \psi_x\psi_{xyy} - 2\psi_{xy}^2 - \psi_{yy}(\psi_{yy} - \psi_{xx})\right]\right], \tag{18}$$

$$S_{xy}+w_1\Big[S_{xy,t}+\psi_y S_{xy,x}-\psi_x S_{xy,y}-\psi_{yy}S_{yy}+\psi_{xx}S_{xx}\Big]=$$
$$2w_2\psi_{xy}\nabla^2\psi+\left(1+w_2\left(\frac{\partial}{\partial t}+\psi_y\frac{\partial}{\partial x}-\psi_x\frac{\partial}{\partial y}\right)\right)(\psi_{yy}-\psi_{xx}), \quad (19)$$

$$S_{yy}+w_1\Big[S_{yy,t}+\psi_y S_{yy,x}-\psi_x S_{yy,y}+2\psi_{xx}S_{xy}+2\psi_{xy}S_{yy}\Big]=$$
$$-2\Big[\psi_{xy}+w_2\Big[\psi_{xyt}+\psi_y\psi_{xxy}-\psi_x\psi_{xyy}+2\psi_{xy}^2-\psi_{xx}(\psi_{yy}-\psi_{xx})\Big]\Big], \quad (20)$$

$$\eta=\varepsilon\ \cos\alpha(x-t), \quad (21)$$

with

$$\psi_y=0,\ \psi_x=\mp\alpha\varepsilon\sin\alpha(x-t)\ \text{ at }\ y=\pm1\pm\eta, \quad (22)$$

where ∇^2 denotes the Laplacian operator and the subscripts indicate partial differentiation.

3. Method of Solution

If the parameter ε is assumed to be small it can be used to perturb the governing equations; see e. g. Van Dyke [21] and Nayfeh [22]. A perturbation solution valid for $\varepsilon \ll 1$ is constructed as

$$\psi=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\ldots, \quad (23)$$

$$\left(\frac{\partial p}{\partial x}\right)=\left(\frac{\partial p}{\partial x}\right)_0+\varepsilon\left(\frac{\partial p}{\partial x}\right)_1+\varepsilon^2\left(\frac{\partial p}{\partial x}\right)_2+\ldots, \quad (24)$$

$$S_{xx}=S_{xx0}+\varepsilon S_{xx1}+\varepsilon^2 S_{xx2}+\ldots, \quad (25)$$

$$S_{xy}=S_{xy0}+\varepsilon S_{xy1}+\varepsilon^2 S_{xy2}+\ldots, \quad (26)$$

$$S_{yy}=S_{yy0}+\varepsilon S_{yy1}+\varepsilon^2 S_{yy2}+\ldots. \quad (27)$$

The first term on the right-hand side of (24) corresponds to the imposed pressure gradient associated with the primary flow, and the other terms correspond to the peristaltic motion. Substituting (23) – (27) into (17) – (20) and (22), and collecting terms like powers of ε, we obtain three sets of coupled differential equations with their corresponding boundary conditions in ε_0, ε_1, and ε_2. The first set of differential equations in ε_0 represents the steady parallel flow and transverse symmetry assumption for a constant pressure gradient in the x-direction under the effect of a transverse magnetic field:

$$\psi_0=\frac{2\,K}{\Gamma^2}\left(y-\frac{\sinh\Gamma y}{\Gamma\cosh\Gamma}\right), \quad (28)$$

$$\Gamma=M\sqrt{R},\quad K=-\frac{R}{2}\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)_0. \quad (29)$$

It can easily be shown that $\psi_0\to K\left(y-\frac{y^3}{3}\right)$ as $M\to0$, which is the classical Poiseuille flow in the absence of a magnetic field which agrees with the work of Fung and Yih [17]. The last solution (28) is independent of the viscoelastic parameters; this means that the flow at this order is independent of the viscoelastic parameters. The second and third sets of differential equations in ψ_1 and ψ_2 with their corresponding boundary conditions are satisfied by

$$\psi_1(x,y,t)=\frac12\left(\phi_1(y)\mathrm{e}^{\mathrm{i}\alpha(x-t)}+\phi_1^*(y)\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right), \quad (30)$$

$$S_{xx1}(x,y,t)=\frac12\left(\phi_2(y)\mathrm{e}^{\mathrm{i}\alpha(x-t)}+\phi_2^*(y)\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right), \quad (31)$$

$$S_{xy1}(x,y,t)=\frac12\left(\phi_3(y)\mathrm{e}^{\mathrm{i}\alpha(x-t)}+\phi_3^*(y)\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right), \quad (32)$$

$$S_{yy1}(x,y,t)=\frac12\left(\phi_4(y)\mathrm{e}^{\mathrm{i}\alpha(x-t)}+\phi_4^*(y)\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right), \quad (33)$$

$$\psi_2(x,y,t)=\frac12\Big(\phi_{20}(y)+\phi_{22}(y)\mathrm{e}^{2\mathrm{i}\alpha(x-t)}+\phi_{22}^*(y)\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big), \quad (34)$$

$$S_{xx2}(x,y,t)=\frac12\Big(\phi_{30}(y)+\phi_{33}(y)\mathrm{e}^{2\mathrm{i}\alpha(x-t)}+\phi_{33}^*(y)\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big), \quad (35)$$

$$S_{xy2}(x,y,t)=\frac12\Big(\phi_{40}(y)+\phi_{44}(y)\mathrm{e}^{2\mathrm{i}\alpha(x-t)}+\phi_{44}^*(y)\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big), \quad (36)$$

$$S_{yy2}(x,y,t)=\frac12\Big(\phi_{50}(y)+\phi_{55}(y)\mathrm{e}^{2\mathrm{i}\alpha(x-t)}+\phi_{55}^*(y)\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big), \quad (37)$$

where, the asterisk denotes the complex conjugate. Substituting (30) – (37) into the differential equations and their corresponding boundary conditions in ψ_1 and ψ_2 we get:

$$\mathrm{i}\alpha R\left(1-\frac{2K}{\Gamma^2}\left(1-\frac{\cosh\Gamma y}{\cosh\Gamma}\right)\right)(\phi_1''-\alpha^2\phi_1)-\left(\frac{2\mathrm{i}\alpha RK\cosh\Gamma y}{\cosh\Gamma}\right)\phi_1=\mathrm{i}\alpha\phi_4'-\mathrm{i}\alpha\phi_2'-\phi_3''-\alpha^2\phi_3+RM^2\phi_1'', \quad (38)$$

$$\begin{aligned}&\cosh\Gamma\left[(1-\mathrm{i}\alpha w_1)\Gamma^2\cosh\Gamma+2\mathrm{i}\alpha w_1K(\cosh\Gamma-\cosh\Gamma y)\right]\phi_2=\\&\left[16\mathrm{i}\alpha w_1K^2(w_1-w_2)\Gamma\sinh\Gamma y\cosh\Gamma y+4\alpha^2w_2K\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_1\\&+2\mathrm{i}\alpha\left[(1-\mathrm{i}\alpha w_2)\Gamma^2\cosh^2\Gamma+8K^2w_1(w_1-w_2)\sinh^2\Gamma y+2\mathrm{i}\alpha Kw_2\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\right]\phi_1'\\&-\left[4Kw_1\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_3+\left[4K(2w_2-w_1)\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_1'',\end{aligned} \quad (39)$$

$$\begin{aligned}&\cosh\Gamma\left[(1-\mathrm{i}\alpha w_1)\Gamma^2\cosh\Gamma+2\mathrm{i}\alpha w_1K(\cosh\Gamma-\cosh\Gamma y)\right]\phi_3=\\&\left[(1-\mathrm{i}\alpha w_2)\Gamma^2\cosh^2\Gamma+2\mathrm{i}\alpha w_2K\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\right]\phi_1''-\left[2w_1K\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_4\\&+\Big[(\alpha^2+\mathrm{i}\alpha^3w_2)\Gamma^2\cosh^2\Gamma+8\alpha^2K^2w_1(w_1-w_2)\sinh^2\Gamma y+2\mathrm{i}\alpha(w_2-w_1)K\Gamma^2\cosh\Gamma\cosh\Gamma y\\&+2\mathrm{i}\alpha^3Kw_2\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\Big]\phi_1-\left[2\mathrm{i}\alpha w_2K\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_1',\end{aligned} \quad (40)$$

$$\begin{aligned}&\left[(1-\mathrm{i}\alpha w_1)\Gamma^2\cosh\Gamma+2\mathrm{i}\alpha w_1K(\cosh\Gamma-\cosh\Gamma y)\right]\phi_4=\\&-\left[4\alpha^2K\Gamma(w_1-w_2)\sinh\Gamma y\right]\phi_1-2\mathrm{i}\alpha\left[(1-\mathrm{i}\alpha w_2)\Gamma^2\cosh\Gamma+2\mathrm{i}\alpha w_2\,K(\cosh\Gamma-\cosh\Gamma y)\right]\phi_1',\end{aligned} \quad (41)$$

with

$$\phi_1(\pm1)=\pm1, \quad \phi_1'(\pm1)=2K, \quad (42)$$

and

$$\phi_{40}''-RM^2\phi_{20}''=\frac{\mathrm{i}\alpha R}{2}\left(\phi_1^*\phi_1''-\phi_1\phi_1^{*\prime\prime}\right)', \quad (43)$$

$$\begin{aligned}&\left[2\Gamma\cosh\Gamma\right]\phi_{30}=\mathrm{i}\alpha w_1\Gamma\cosh\Gamma\left[3\phi_1'\phi_2^*-3\phi_2\phi^{*\prime}_1-\phi_2'\phi_1^*+\phi_1\phi_2^{*\prime}\right]\\&-4w_2\Gamma\cosh\Gamma\left[\alpha^2(\phi_1\phi_1^{*\prime\prime}+\phi_1''\phi_1^*+3\phi_1'\phi_1^{*\prime})+\phi_1''\phi_1^{*\prime\prime}\right]-\left[8Kw_1\sinh\Gamma y\right]\phi_{40}\\&-2w_1\Gamma\cosh\Gamma\left[\phi_3\phi_1^{*\prime\prime}+\phi_3^*\phi_1''\right]-\left[8K(w_1-2w_2)\sinh\Gamma y\right]\phi_{20}'',\end{aligned} \quad (44)$$

$$\begin{aligned}&2\Gamma\cosh\Gamma\left[\phi_{40}-\phi_{20}''\right]=w_1\Gamma\cosh\Gamma\left[\phi_4\phi_1^{*\prime\prime}+\phi_4^*\phi_1''+\alpha^2(\phi_1\phi_2^*+\phi_2\phi_1^*)\right]\\&+2\mathrm{i}\alpha^3w_2\Gamma\cosh\Gamma\left[\phi_1\phi_1^{*\prime}-\phi_1'\phi_1^*\right]-\left[4Kw_1\sinh\Gamma y\right]\phi_{50}\\&-\mathrm{i}\alpha\Gamma\cosh\Gamma\left[w_2(\phi_1\phi_1^{*\prime\prime}-\phi_1''\phi_1^*)+w_1(\phi_3\phi_1^*-\phi_1\phi_3^*)\right]',\end{aligned} \quad (45)$$

$$2\phi_{50}=\mathrm{i}\alpha w_1\left[\phi_4\phi_1^*-\phi_4^*\phi_1\right]'+2\alpha^2w_1\left[\phi_1\phi_3^*-\phi_1^*\phi_3\right]-4\alpha^2w_2\left[\phi_1'\phi_1^{*\prime}+\alpha^2\phi_1\phi_1^*\right], \quad (46)$$

with

$$\phi_{20}'(\pm\,1)=K\mp\frac{1}{2}\left(\phi_1''(\pm\,1)+\phi_1^{*\prime\prime}(\pm\,1)\right), \quad (47)$$

and

$$4\alpha R\left[1-\frac{2K}{\Gamma^2}\left(1-\frac{\cosh\Gamma y}{\cosh\Gamma}\right)\right]\left[\phi_{22}''-4\alpha^2\phi_{22}\right]-\frac{8\alpha RK\cosh\Gamma y}{\cosh\Gamma}\phi_{22}= \alpha R\left[\phi_1'\phi_1''-\phi_1\phi_1'''\right]-4\alpha\phi_{33}'+2\mathrm{i}\phi_{44}''+8\mathrm{i}\alpha^2\phi_{44}+4\alpha\phi_{55}'-2\mathrm{i}RM^2\phi_{22}'', \quad (48)$$

$$2\cosh\Gamma\left[\Gamma^2\cosh\Gamma-2\mathrm{i}\alpha w_1(\Gamma^2\cosh\Gamma-2K(\cosh\Gamma-\cosh\Gamma y))\right]\phi_{33}= \Gamma^2\cosh^2\Gamma\left[\mathrm{i}\alpha w_1(\phi_1\phi_2'+\phi_1'\phi_2)-2w_1\phi_3\phi_1''\right] +8\mathrm{i}\alpha\left[\Gamma^2\cosh^2\Gamma+8K^2w_1(w_1-w_2)\sinh^2\Gamma y-2\mathrm{i}\alpha w_2(\Gamma^2\cosh^2\Gamma-2K(\cosh^2\Gamma-\cosh\Gamma\cosh\Gamma y))\right]\phi_{22}' -8K(w_1-w_2)\left[\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_{22}''+2w_2\Gamma^2\cosh^2\Gamma\left[\alpha^2\phi_1'^2-\phi_1''^2\right]-\left[8Kw_1\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_{44} +4\alpha\left[16\mathrm{i}K^2\Gamma w_1(w_1-w_2)\sinh\Gamma y\cosh\Gamma y+8\alpha K\Gamma w_2\cosh\Gamma\sinh\Gamma y\right]\phi_{22}, \quad (49)$$

$$2\cosh\Gamma\left[\Gamma^2\cosh\Gamma-2\mathrm{i}\alpha w_1(\Gamma^2\cosh\Gamma-2K(\cosh\Gamma-\cosh\Gamma y))\right]\phi_{44}= w_1\Gamma^2\cosh^2\Gamma\left[\mathrm{i}\alpha(\phi_1\phi_3'-\phi_1'\phi_3)+\phi_4\phi_1''+\alpha^2\phi_1\phi_2\right]+\mathrm{i}\alpha w_2\Gamma^2\cosh^2\Gamma\left[3\phi_1'\phi_1''-\phi_1\phi_1'''-2\alpha^2\phi_1\phi_1'\right] -\left[16\mathrm{i}\alpha Kw_2\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_{22}'-\left[4Kw_1\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_{55} +\left[(2-4\mathrm{i}\alpha w_2)\Gamma^2\cosh^2\Gamma-16\mathrm{i}\alpha Kw_2\Gamma\cosh\Gamma\sinh\Gamma y+8\mathrm{i}\alpha Kw_2\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\right]\phi_{22}'' +8\mathrm{i}\alpha\left[8\alpha K^2w_1(w_1-w_2)\sinh^2\Gamma y-Kw_1\Gamma^2\cosh\Gamma\cosh\Gamma y-\mathrm{i}(1-2\mathrm{i}\alpha w_2)\alpha\Gamma^2\cosh^2\Gamma +Kw_2\Gamma^2\cosh\Gamma\cosh\Gamma y+4\alpha^2Kw_2\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\right]\phi_{22}, \quad (50)$$

$$2\cosh\Gamma\left[\Gamma^2\cosh\Gamma-2\mathrm{i}\alpha w_1(\Gamma^2\cosh\Gamma-2K(\cosh\Gamma-\cosh\Gamma y))\right]\phi_{55}= \Gamma^2\cosh^2\Gamma\left[2\alpha^2w_2(3\phi_1'^2-2\phi_1\phi_1''-\alpha^2\phi_1^2)+\mathrm{i}\alpha w_1(\phi_1\phi_4'-3\phi_1'\phi_4-2\mathrm{i}\alpha\phi_1\phi_3)\right]-\left[8\mathrm{i}\alpha\Gamma^2\cosh^2\Gamma\right]\phi_{22}' -\left[32\alpha^2K(w_1-w_2)\Gamma\cosh\Gamma\sinh\Gamma y\right]\phi_{22}-16\alpha^2w_2\left[\Gamma^2\cosh^2\Gamma-2K\cosh\Gamma(\cosh\Gamma-\cosh\Gamma y)\right]\phi_{22}', \quad (51)$$

with

$$\phi_{22}(\pm 1)=\mp\frac{1}{4}\phi_1'(\pm 1), \quad (52)$$

$$\phi_{22}'(\pm 1)=\frac{K}{2}\mp\frac{1}{2}\phi_1''(\pm 1), \quad (53)$$

where $(')$ denotes the derivative with respect to y. These equations are sufficient to determine the solution up to the second order in ε. But these equations are fourth-order ordinary differential equations with variable coefficients, and the boundary conditions are not all homogeneous and the problem is no eigenvalue problem. However, we can restrict our investigation to the case of free-pumping. Physically, this means that the fluid is stationary if there are no peristaltic waves. In this case we put $\left(\frac{\partial p}{\partial x}\right)_0=0$ which means $K=0$. Under this assumption we get a solutions of (38)–(41) in the form

$$\phi_1(y) = A1\sinh\alpha_1 y+B1\sinh\beta_1 y, \quad (54)$$

$$\phi_1(y) = A1\sinh\alpha_1 y+B1\sinh\beta_1 y, \quad (55)$$

$$\phi_3(y) = A3\sinh\alpha_1 y+B3\sinh\beta_1 y, \quad (56)$$

$$\phi_4(y) = -A2\cosh\alpha_1 y-B2\cosh\beta_1 y, \quad (57)$$

where

$$A1 = \frac{-\beta_1 \cosh\beta_1}{\alpha_1 \cosh\alpha_1 \sinh\beta_1 - \beta_1 \cosh\beta_1 \sinh\alpha_1}, \quad (58)$$

$$B1 = \frac{\alpha_1 \cosh\alpha_1}{\alpha_1 \cosh\alpha_1 \sinh\beta_1 - \beta_1 \cosh\beta_1 \sinh\alpha_1}, \quad (59)$$

$$A2 = 2\mathrm{i}\alpha\alpha_1 E A1, \quad B2 = 2\mathrm{i}\alpha\beta_1 E B1, \quad (60)$$

$$A3 = (\alpha^2 + \alpha_1^2) E A1, \quad B3 = (\alpha^2 + \beta_1^2) E B1, \quad (61)$$

$$\beta_1^2 = \frac{\Gamma^2 + 2\alpha^2 E - \mathrm{i}\alpha R - \sqrt{\Gamma^4 - \alpha^2 R^2 - 2\mathrm{i}\alpha R\Gamma^2 + 4\alpha^2\Gamma^2 E}}{2E}, \quad (62)$$

$$\alpha_1^2 = \frac{\Gamma^2 + 2\alpha^2 E - \mathrm{i}\alpha R + \sqrt{\Gamma^4 - \alpha^2 R^2 - 2\mathrm{i}\alpha R\Gamma^2 + 4\alpha^2\Gamma^2 E}}{2E}, \quad (63)$$

$$E = \frac{(1 - \mathrm{i}\alpha w_2)(1 + \mathrm{i}\alpha w_1)}{(1 + \alpha^2 w_1^2)}. \quad (64)$$

Next, in the expansion of ψ_2, we must only concern ourselves with the terms $\phi'_{20}(y)$, as our aim is to determine the mean flow only. Thus, the differential equations (43)–(46) subjected to the boundary condition (47), under the assumption $K = 0$, give the expression

$$\phi'_{20}(y) = F(y) + 2C1 \frac{\cosh(\Gamma y) - \cosh(\Gamma)}{\Gamma^2 \cosh(\Gamma)} + (D - F(1)) \frac{\cosh(\Gamma y)}{\cosh(\Gamma)}, \quad (65)$$

were

$$D = \phi'_{20}(\pm 1) = -\frac{1}{2}\left[\alpha_1^2 A1 \sinh\alpha_1 + \alpha_1^{*2} A1^* \sinh\alpha_1^* + \beta_1^2 B1 \sinh\beta_1 + \beta_1^{*2} B1^* \sinh\beta_1^*\right], \quad (66)$$

$$\begin{aligned} F(y) = {} & s_1 \cosh(\alpha_1 + \beta_1^*)y + s_2 \cosh(\alpha_1 - \beta_1^*)y + s_3 \cosh(\alpha_1^* + \beta_1)y + s_4 \cosh(\alpha_1^* - \beta_1)y \\ & + s_5 \cosh(\beta_1 + \beta_1^*)y + s_6 \cosh(\beta_1 - \beta_1^*)y + s_7 \cosh(\alpha_1 + \alpha_1^*)y + s_8 \cosh(\alpha_1 - \alpha_1^*)y, \end{aligned} \quad (67)$$

$$\begin{aligned} s_1 = {} & \frac{(\alpha_1 + \beta_1^*)}{4((\alpha_1 + \beta_1^*)^2 - \Gamma^2)} \Big[w_1(\alpha_1^2 - \alpha^2) A1 B2^* + \mathrm{i}\alpha w_1(\alpha_1 + \beta_1^*)(A3 B1^* - A1 B3^*) \\ & + w_1(\beta_1^{*2} - \alpha^2) A2 B1^* + \mathrm{i}\alpha(\alpha_1 - \beta_1^*)(R + 2\alpha^2 w_2 - w_2(\alpha_1 + \beta_1^*)^2) A1 B1^* \Big], \end{aligned} \quad (68)$$

$$\begin{aligned} s_2 = {} & \frac{(\alpha_1 - \beta_1^*)}{4((\alpha_1 - \beta_1^*)^2 - \Gamma^2)} \Big[w_1(\alpha_1^2 - \alpha^2) A1 B2^* - \mathrm{i}\alpha w_1(\alpha_1 - \beta_1^*)(A3\, B1^* - A1 B3^*) \\ & + w_1(\beta_1^{*2} - \alpha^2) A2 B1^* - \mathrm{i}\alpha(\alpha_1 + \beta_1^*)(R + 2\alpha^2 w_2 + w_2(\alpha_1 - \beta_1^*)^2) A1 B1^* \Big], \end{aligned} \quad (69)$$

$$\begin{aligned} s_3 = {} & \frac{(\alpha_1^* + \beta_1)}{4((\alpha_1^* + \beta_1)^2 - \Gamma^2)} \Big[w_1(\alpha_1^{*2} - \alpha^2) B2 A1^* + \mathrm{i}\alpha w_1(\beta_1 + \alpha_1^*)(B3\, A1^* - B1\, A3^*) \\ & + w_1(\beta_1^2 - \alpha^2) B1 A2^* + \mathrm{i}\alpha(\beta_1 - \alpha_1^*)(R + 2\alpha^2 w_2 - w_2(\alpha_1^* + \beta_1)^2) B1 A1^* \Big], \end{aligned} \quad (70)$$

$$\begin{aligned} s_4 = {} & \frac{(\alpha_1^* - \beta_1)}{4((\alpha_1^* - \beta_1)^2 - \Gamma^2)} \Big[w_1(\alpha_1^{*2} - \alpha^2) B2 A1^* + \mathrm{i}\alpha w_1(\beta_1 - \alpha_1^*)(B3 A1^* - B1 A3^*) \\ & + w_1(\alpha^2 - \beta_1^2) B1 A2^* + \mathrm{i}\alpha(\alpha_1^* + \beta_1)(R + 2\alpha^2 w_2 - w_2(\alpha_1^* - \beta_1)^2) B1 A1^* \Big], \end{aligned} \quad (71)$$

$$\begin{aligned} s_5 = {} & \frac{(\beta_1 + \beta_1^*)}{4((\beta_1 + \beta_1^*)^2 - \Gamma^2)} \Big[w_1(\beta_1^2 - \alpha^2) B1 B2^* + \mathrm{i}\alpha w_1(\beta_1 + \beta_1^*)(B3 B1^* - B1 B3^*) \\ & + w_1(\beta_1^{*2} - \alpha^2) B2 B1^* - \mathrm{i}\alpha(\beta_1 - \beta_1^*)(R + 2\alpha^2 w_2 - w_2(\beta_1 + \beta_1^*)^2) B1 B1^* \Big], \end{aligned} \quad (72)$$

$$s_6 = \frac{(\beta_1-\beta_1^*)}{4((\beta_1-\beta_1^*)^2-\Gamma^2)}\Big[w_1(\beta_1^2-\alpha^2)B1B2^* - \mathrm{i}\alpha w_1(\beta_1-\beta_1^*)(B3B1^*-B1B3^*) + w_1(\alpha^2-\beta_1^{*2})B2B1^* - \mathrm{i}\alpha(\beta_1+\beta_1^*)(R+2\alpha^2 w_2 - w_2(\beta_1-\beta_1^*)^2)B1B1^*\Big], \quad (73)$$

$$s_7 = \frac{(\alpha_1+\alpha_1^*)}{4((\alpha_1+\alpha_1^*)^2-\Gamma^2)}\Big[w_1(\alpha_1^2-\alpha^2)A1A2^* + \mathrm{i}\alpha w_1(\alpha_1+\alpha_1^*)(A3A1^*-A1A3^*) + w_1(\alpha_1^{*2}-\alpha^{*2})A2A1^* - \mathrm{i}\alpha(\alpha_1^*-\alpha_1)(R+2\alpha^2 w_2 - w_2(\alpha_1^*+\alpha_1)^2)A1A1^*\Big], \quad (74)$$

$$s_8 = \frac{(\alpha_1-\alpha_1^*)}{4((\alpha_1-\alpha_1^*)^2-\Gamma^2)}\Big[w_1(\alpha_1^2-\alpha^2)A1A2^* + \mathrm{i}\alpha w_1(\alpha_1^*-\alpha_1)(A3A1^*-A1A3^*) + w_1(\alpha^2-\alpha_1^{*2})A2A1^* - \mathrm{i}\alpha(\alpha_1+\alpha_1^*)(R+2\alpha^2 w_2 - w_2(\alpha_1-\alpha_1^*)^2)A1A1^*\Big]. \quad (75)$$

Thus, we see that one constant, $C1$, remains arbitrary in the solution. Substituting (23)–(27) into (10), and time-averaging the equation of second-order of ε with the assumption $K = 0$, we find

$$C1 = R\left(\frac{\partial \bar{p}}{\partial x}\right)_2. \quad (76)$$

Also, the mean time-averaged velocity may be written as

$$\bar{u}(y) = \frac{\varepsilon^2}{2}\phi'_{20}(y) = \frac{\varepsilon^2}{2}\Big[F(y)+(D-F(1))\frac{\cosh\Gamma y}{\cosh\Gamma} + \frac{2R}{\Gamma^2}\left(\frac{\partial \bar{p}}{\partial x}\right)_2\left(\frac{\cosh\Gamma y-\cosh\Gamma}{\cosh\Gamma}\right)\Big]. \quad (77)$$

Note that, if we put the magnetic parameter M and the Weissenberg numbers w_1 and w_2 equal to zero, then the results of the problem reduce exactly to the same as that found by Fung and Yih [17] for a Newtonian fluid.

4. Numerical Results and Discussion

In order to observe the quantitative effects of various parameters involved in the analysis, the mean-velocity at the boundaries of the channel, the time-averaged mean axial-velocity distribution and the reversal flow are calculated for various values of these parameters in the free pumping case. Numerical calculations based on (77) show that the mean axial-velocity of the fluid due to peristaltic motion is dominated by the constant D and the term $\frac{2R}{\Gamma^2}\left(\frac{\partial \bar{p}}{\partial x}\right)_2\left(\frac{\cosh(\Gamma y)-\cosh(\Gamma)}{\cosh(\Gamma)}\right)$. In addition to these terms, there is a perturbation term $F(y)-F(1)\frac{\cosh(\Gamma y)}{\cosh(\Gamma)}$, which controls the direction of the peristaltic mean flow across the cross-section. The constant D, which initially arose from the non-slip condition of the axial-velocity on the wall, is due to the value of $\phi'_{20}(y)$ at the boundary and is related to the mean-velocity at the boundaries of the channel by $\bar{u}(\pm 1) = \frac{\varepsilon^2}{2}\phi'_{20}(\pm 1) = \frac{\varepsilon^2}{2}D$. Figures 2 and 3 represent the variation of D with the wave number α for various values of the magnetic parameter M and the Weissenberg number w_1. The numerical results indicate that D decreases with increasing w_1 and increases with increasing M and α. Yin and Fung [23] defined a flow reflux whenever there is a negative mean-velocity in the flow field. Then, according to (77), the critical reflux condition is given by $\left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}} = \frac{\Gamma^2}{2R(1-\cosh(\Gamma))}(F(1)-F(0)\cosh(\Gamma)-D)$, and the reflux occurs when $\left(\frac{\partial \bar{p}}{\partial x}\right)_2 > \left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}}$. Figures 4 and 5 represent the variation of $\left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}}$ with α for various values of M and w_1. The results reveal that $\left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}}$ decreases with increasing M and w_1. The effects of M and w_1 on the mean-velocity and reversal flow are displayed in Figs. 6 and 7. The results reveal that the reversal flow increases with increasing M and w_1. Similar results were obtained for the Weissenberg number w_2 and are not presented here. The analysis shows that the zeroth-order solution has been found to be identical to that for Newtonian behavior. At this order, it is found that the Weissenberg numbers only contribute to S_{xx0}. Higher-order solutions have been studied to reveal the effects of non-Newtonian behavior on peristaltic waves. The results indicate that the second-order solution depends strongly on the Weissenberg numbers.

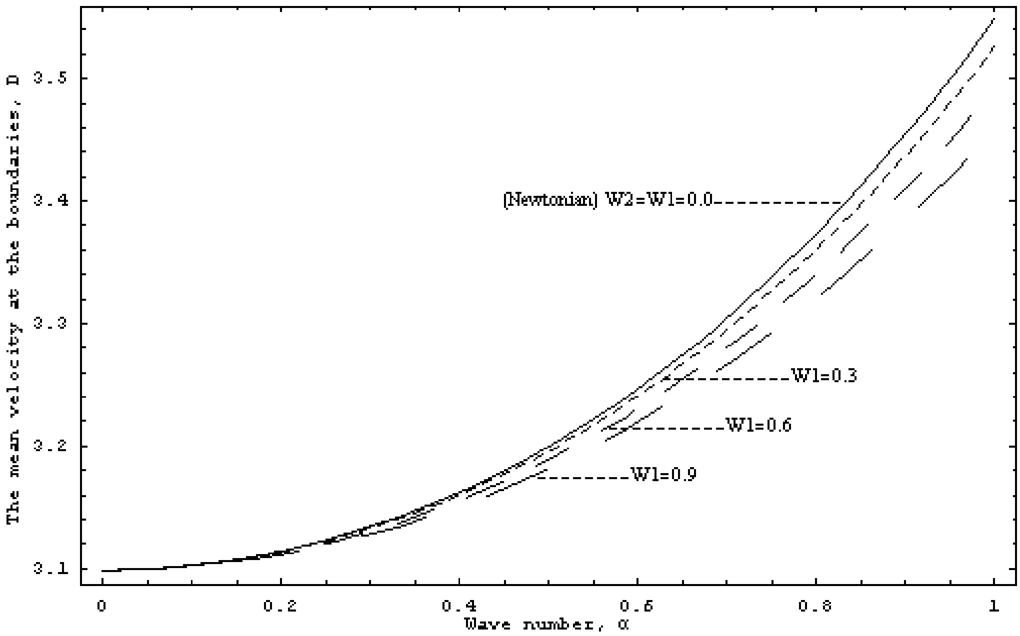

Fig. 2. Effect of the Weissenberg number w_1 on the variation of D with the wave number α for $w_2 = 0.1$, $M = 1$, and $R = 0.5$.

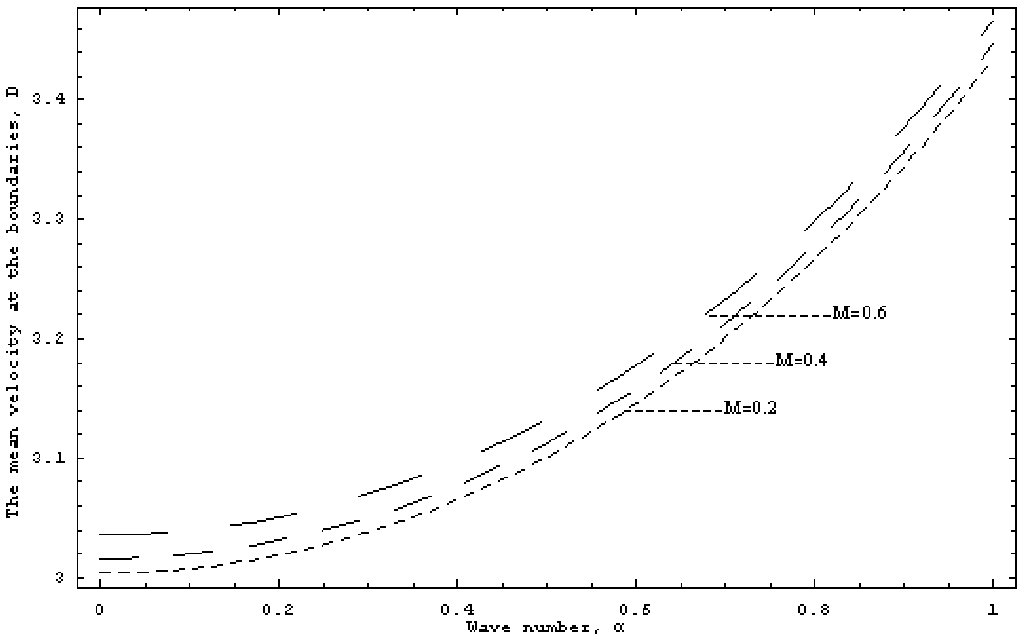

Fig. 3. Effect of the magnetic parameter M on the variation of D with the wave number α for $w_1 = 0.8$, $w_2 = 0.5$, and $R = 0.5$.

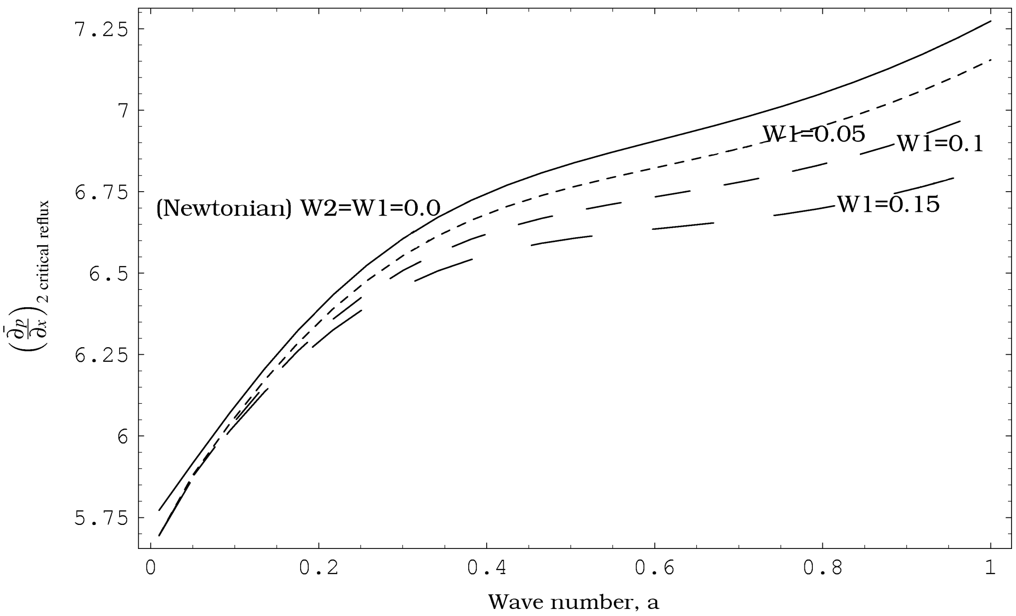

Fig. 4. Effect of the Weissenberg number w_1 on the variation of the critical reflux pressure gradient $\left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}}$ with the wave number α for $w_2 = 0.001$, $M = 2$, and $R = 0.5$.

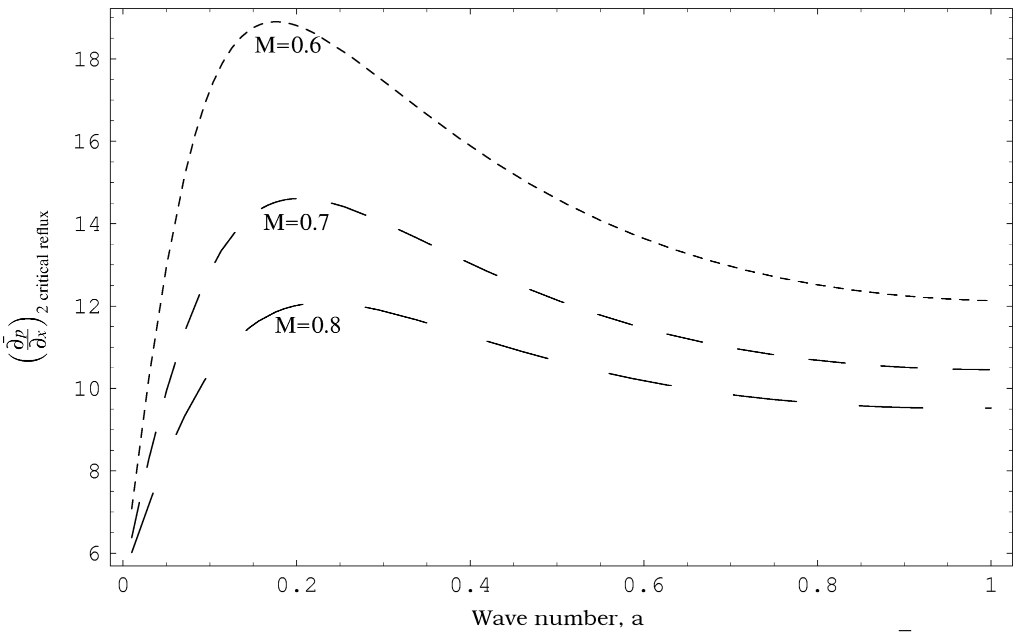

Fig. 5. Effect of the magnetic parameter M on the variation of the critical reflux pressure gradient $\left(\frac{\partial \bar{p}}{\partial x}\right)_{2\text{ critical reflux}}$ with the wave number α for $w_1 = 0.1$, $w_2 = 0.01$, and $R = 0.5$.

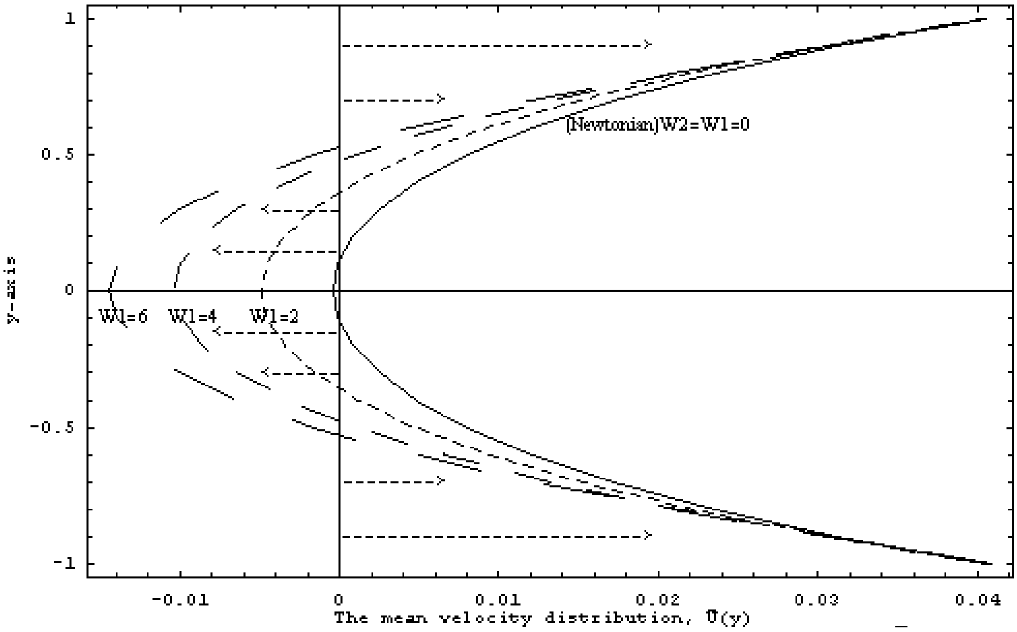

Fig. 6. Effect of the Weissenberg number w_1 on the mean-velocity distribution and reversal flow for $\left(\frac{\partial\bar{p}}{\partial x}\right)_2 = 4$, $w_2 = 0.01$, $\alpha = 0.2$, $M = 2$, $\varepsilon = 0.15$, and $R = 0.8$.

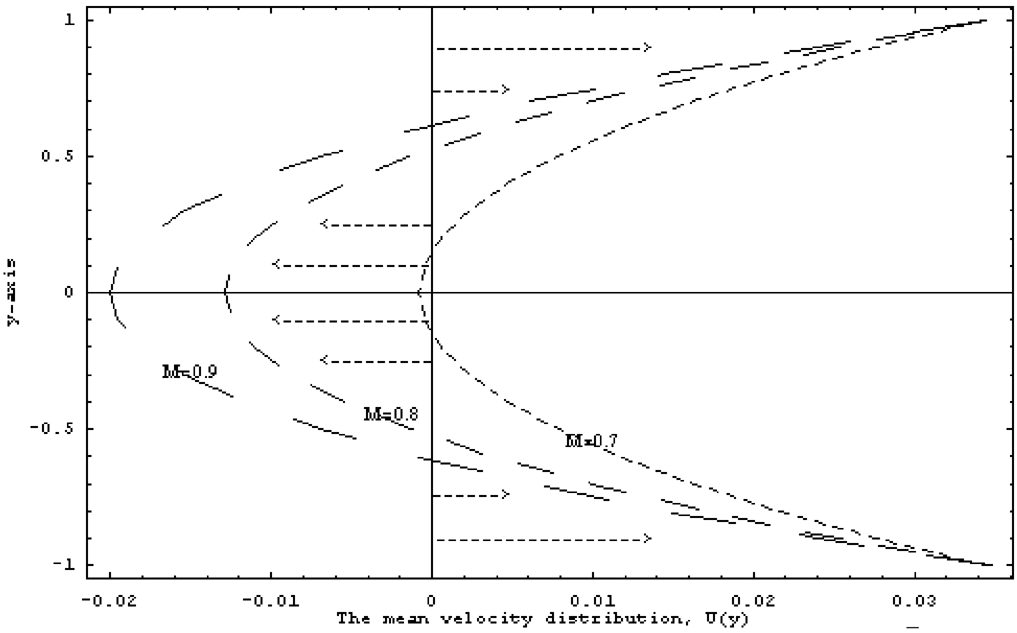

Fig. 7. Effect of the magnetic parameter M on the mean-velocity distribution and reversal flow for $\left(\frac{\partial\bar{p}}{\partial x}\right)_2 = 8$, $w_1 = 0.8$, $w_2 = 0.5$, $\alpha = 0.2$, $\varepsilon = 0.15$, and $R = 0.5$.

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